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Ratio Estimation with Measurement Error in the Auxiliary Variate

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Summary. With auxiliary information that is well correlated with the primary variable of interest, ratio estimation of the finite population total may be much more efficient than alternative estimators that do not make use of the auxiliary variate. The well-known properties of ratio estimators are perturbed when the auxiliary variate is measured with error. In this contribution we examine the effect of measurement error in the auxiliary variate on the design-based statistical properties of three common ratio estimators. We examine the case of systematic measurement error as well as measurement error that varies according to a fixed distribution. Aside from presenting expressions for the bias and variance of these estimators when they are contaminated with measurement error we provide numerical results based on a specific population. Under systematic measurement error, the biasing effect is asymmetric around zero, and precision may be improved or degraded depending on the magnitude of the error. Under variable measurement error, bias of the conventional ratio-of-means estimator increased slightly with increasing error dispersion, but far less than the increased bias of the conventional mean-of-ratios estimator. In similar fashion, the variance of the mean-of-ratios estimator incurs a greater loss of precision with increasing error dispersion compared with the other estimators we examine. Overall, the ratio-of-means estimator appears to be remarkably resistant to the effects of measurement error in the auxiliary variate.

Key words: Beta errors; Design-based inference; Gaussian errors; Rounding errors; Sampling; Surveys; Uniform errors.

1. Introduction

In the field of survey sampling, auxiliary information is used variously to guide the inclusion of population units into the sample or to assist in the estimation of population descriptive parameters, or both. Classic, recent, and contemporary texts on sampling—for example, Cochran (1977), Särndal, Swensson, and Wretman (1992), and Gregoire and Valentine (2008)—all devote substantial portions of their expositions to the use of auxiliary information to increase the precision with which quantitative population characteristics may be estimated. To be worthwhile for these purposes, the auxiliary variate, say x, must be well correlated, usually positively correlated, with the attribute of principal interest, say y, and it must be comparatively inexpensive to obtain, or else the investment in time and energy to obtain information on the auxiliary variate could be better spent obtaining a larger sample of the variable of interest alone.

Oftentimes the secondary importance of the auxiliary variate, x, may result in a more sizeable error in its measurement compared to that of y. For example, x may result from remotely sensed, satellite data that have been processed by a digital classifier, and hence subject to classification error. Or, x may be an ocular or other type of subjective assessment of a size characteristic that is associated with y. Airborne or space Light Detection and Ranging (LiDAR) measurements of forest canopy height may change due to degradation in laser system performance over time, due to location error between airborne laser profiles, or due to technological improvements that necessitate the use of two different laser systems for data collection flown years apart. Yet another source of error when x is measured in the field or forest is nonnegligible rounding error. These few examples are hardly exhaustive, and they are provided merely to help motivate the problem we address: because of the nature of auxiliary information and how it is used in the sample design and estimation, it is apt to include nonnegligible measurement error.

The purpose of this article is to explore the effect of measurement error (ME) in x on the design-based statistical properties of commonly employed ratio estimators of τ_y, the total amount of the variate of interest, y, in the population. We know of no prior investigation of this issue, notwithstanding considerable recent attention devoted to measurement error in modeling (e.g., Fuller, 1987; Carroll, 1998) and its effect on model-based inference. On one level it is evident that although the information in x about y is degraded when measurement error is included, the well-known properties of ratio estimators persist. However, on a different level we think that there is value in scrutinizing more specific effects of measurement error. For example, whether it affects bias more egregiously than variance; whether the relative impact changes with sample size; and whether the effects are symmetric around zero. The results of this article shed light on such effects.

2. Ratio Estimation of Population Descriptive Parameters

2.1 *Population Descriptive Parameters*

Similar to Breidt et al. (2007), we denote the ordered labels for a finite population of discrete units by $\mathcal{P} = \{1, 2, \ldots N\}$, indexed by k in the following. Corresponding to each unit is a nonnegative value of some measurable characteristic, $y_k, \forall\, k \in \mathcal{P}$; likewise, there is a quantitative, auxiliary characteristic, x_k, also nonnegative. Our interest is focused on the estimation of the aggregate value of y in $\mathcal{P}$, namely, $\tau_y = \sum_{\mathcal{P}} y_k$ on the basis of a sample of n units selected without replacement by simple random sampling (SRSwoR). We denote the mean y-value per element in $\mathcal{P}$ by $\mu_y = \tau_y / N$.

2.2 *Ratio Estimation in the Absence of Measurement Error in x*

We first summarize the design-based bias and variance of three ratio estimators of interest when the auxiliary variate is free of measurement error. In the succeeding section, we examine how these properties are altered by measurement error.

2.2.1 *Ratio-of-means estimator.* Letting $\overline{y}$ and $\overline{x}$ denote sample mean values, the aptly named "ratio-of-means" estimator is

$$\widehat{\tau}_{y1} = \widehat{R}\tau_x, \tag{1}$$

where $\tau_x = \sum_{\mathcal{P}} x_k$. In (1), $\widehat{R} = \overline{y}/\overline{x}$ is an estimator of $R = \tau_y/\tau_x = \mu_y/\mu_x$.

A Taylor-series expansion of $\widehat{\tau}_{y1}$ truncated after two terms leads to an expression of its design bias as an estimator of τ_y as

$$B\left[\widehat{\tau}_{y1} : \tau_y\right] \approx \left(\frac{1}{n} - \frac{1}{N}\right)\left(\gamma_x^2 - \rho\gamma_x\gamma_y\right)\tau_y, \tag{2}$$

where ρ is the linear correlation coefficient and γ_x and γ_y are the coefficients of variation of x and y, respectively, in the population. This approximation to the bias of $\widehat{\tau}_{y1}$ may be deduced inter alia, from Cochran (1977, Section 6.8, Eq. 6.34).

The usual approximation of the variance of $\widehat{\tau}_{y1}$ following SRSwoR is (cf. Gregoire and Valentine, 2008, Section 6.3, Eq. 6.16)

$$V\left[\widehat{\tau}_{y1}\right] = N^2\left(\frac{1}{n} - \frac{1}{N}\right)\sigma_{rm}^2, \tag{3}$$

where $\sigma_{rm}^2 = \frac{1}{N-1}\sum_{\mathcal{P}}\left(y_k - Rx_k\right)^2$.

2.2.2 *Mean-of-ratios estimator.* In addition to $\widehat{\tau}_{y1}$ we consider the "mean-of-ratios" estimator,

$$\widehat{\tau}_{y2} = \overline{r}\tau_x, \tag{4}$$

where $\overline{r}$ is the average ratio $r_k = y_k/x_k$ of the n units in the sample. Its expected value under the SRSwoR design is $E[\widehat{\tau}_{y2}] = \mu_r\tau_x$, where μ_r is the population average ratio, i.e., $\mu_r = \sum_{\mathcal{P}} r_k/N$.

The design bias of $\widehat{\tau}_{y2}$ as an estimator of τ_y may be deduced straightforwardly as

$$B\left[\widehat{\tau}_{y2} : \tau_y\right] = \sum_{\mathcal{P}} y_k\left(\frac{\mu_x}{x_k} - 1\right).$$

Evidently its bias is impervious to the size of the sample actually selected. Indeed, even when $n = N, \widehat{\tau}_{y2} \neq \tau_y$, and according to Sukhatme and Sukhatme (1970, p. 160), this inconsistency has limited its use. The variance of $\widehat{\tau}_{y2}$ is

$$V\left[\widehat{\tau}_{y2}\right] = \tau_x^2\left(\frac{1}{n} - \frac{1}{N}\right)\sigma_{m\,r}^2,$$

where $\sigma_{m\,r}^2 = \frac{1}{N-1}\sum_{\mathcal{P}}(r_k - \mu_r)^2$, as shown in equation (7) of Goodman and Hartley (1958).

2.2.3 *Unbiased ratio-type estimator.* Hartley and Ross (1954) introduced the following design-unbiased estimator of τ_y:

$$\widehat{\tau}_{y3} = \widehat{\tau}_{y2} + \left(\frac{N-1}{N}\right)\left(\frac{n}{n-1}\right)\left(\widehat{\tau}_{y\pi} - \overline{r}\,\widehat{\tau}_{x\pi}\right),$$

in which $\widehat{\tau}_{y\pi}$ and $\widehat{\tau}_{x\pi}$ are the Horvitz–Thompson estimators of τ_y and τ_x, respectively. Unbiasedness of $\widehat{\tau}_{y3}$ is obtained because $E[(\frac{N-1}{N})(\frac{n}{n-1})(\widehat{\tau}_{y\pi} - \overline{r}\,\widehat{\tau}_{x\pi})] = \tau_y - \mu_r\tau_x$.

Goodman and Hartley (1958, Eq. 18) derived the variance of $\widehat{\tau}_{y3}$. In our notation and after adjusting to a finite population context,

$$V[\widehat{\tau}_{y3}] = N^2\left(\frac{1}{n} - \frac{1}{N}\right)\left(\sigma_y^2 + \mu_r^2\sigma_x^2 - 2\mu_r C(x,y) + \frac{1}{n-1}\left(\sigma_r^2\sigma_x^2 + C(r,x)^2\right)\right),$$

where $C(x,\ y)$ is the population covariance: $C(x,y) = \sum_{\mathcal{P}}(x_k - \mu_x)(y_k - \mu_y)/N$, and $C(r,\ x)$ is analogously defined.

2.3 *Ratio Estimation in the Presence of Additive Measurement Error in x*

We assume that x_k cannot be measured without error, consequently in its stead we measure

$$x_k^* = x_k + \delta_k,$$

where δ_k is the measurement error. In keeping with precepts of design-based inference, we assume further that δ_k is fixed in the sense that repeated measurements of the kth unit of $\mathcal{P}$ would result in the same value x_k^*. Fixed error is reasonable, for example, in the case where a measure of length is rounded to the nearest centimeter: We presume that repeated measurements of the same length would result in an identical measurement, the error of which would be unknown but have constant magnitude among the repeated measurements. In a remote sensing context LiDAR readings of height will contain error, the magnitude of which will vary among pixels, yet for a given scene it will be fixed for each pixel in the scene. As a further example, measurement error of a fixed magnitude may result from faulty instrumentation, thereby leading to the same magnitude of measurement error among all elements of $\mathcal{P}$. Cochran (1977, Section 13.9) terms the latter "constant bias over all units," yet he discusses the case where such biased measurement only affects y, not x.

As explained in the next section, we shall consider both the case where the magnitude of δ_k may vary among units, and when it is constant for all units.

In the sequel, the population and sample mean measurement errors are denoted as μ_δ and $\overline{\delta}$, respectively. In an obvious extension to the notation introduced above, let the error-contaminated total, mean, and coefficient of variation of x^* in $\mathcal{P}$ be denoted by τ_x^*, μ_x^*, and γ_x^*, respectively. In similar fashion, the sample mean of x^* is $\overline{x}^*$.

The analog to $\widehat{\tau}_{y1}$ with the error-contaminated auxiliary variate is

$$\widehat{\tau}_{y1}^* = \widehat{R}^* \tau_x^* = \widehat{\tau}_{y1}\left(1 + \frac{\mu_\delta}{\mu_x}\right) \bigg/ \left(1 + \frac{\overline{\delta}}{\overline{x}}\right),$$

where $\widehat{R}^* = \overline{y}/\overline{x}^* = \widehat{R}/(1 + \frac{\overline{\delta}}{\overline{x}})$. The bias of $\widehat{\tau}_{y1}^*$ is analogous to equation (2):

$$B\left[\widehat{\tau}_{y1}^* : \tau_y\right] \approx \left(\frac{1}{n} - \frac{1}{N}\right)\left(\gamma_x^{*2} - \rho^*\gamma_x^*\gamma_y\right)\tau_y. \tag{5}$$

The change in magnitude of bias, which arises from the measurement error, may be expressed usefully by the ratio of equation (5) to that of equation (2), namely,

$$\frac{B\left[\widehat{\tau}_{y1}^* : \tau_y\right]}{B\left[\widehat{\tau}_{y1} : \tau_y\right]} = \frac{\gamma_x^*}{\gamma_x}\left(\frac{\gamma_x^* - \rho^*\gamma_y}{\gamma_x - \rho\gamma_y}\right) = \left(\frac{\mu_y\sigma_x^{*2} - C(x^*, y)(\mu_x + \mu_\delta)}{\mu_y\sigma_x^2 - C(x, y)\mu_x}\right) \bigg/ \left(1 + \frac{\mu_\delta}{\mu_x}\right)^2. \tag{6}$$

The utility of this expression is that it is independent of sample size, n. It shows, also, that even when the population mean error, μ_δ, is identically zero, the bias of $\widehat{\tau}_{y1}^*$ is affected by the variability of measurement error.

From equation (3) we deduce that the approximate variance of $\widehat{\tau}_{y1}^*$ is given by

$$V\left[\widehat{\tau}_{y1}^*\right] = N^2\left(\frac{1}{n} - \frac{1}{N}\right)\sigma_{rm}^{*2}, \tag{7}$$

where

$$\sigma_{rm}^{*2} = \frac{1}{N-1}\sum_{\mathcal{P}}\left(y_k - R^* x_k^*\right)^2 = \frac{1}{N-1}\sum_{\mathcal{P}}\left(y_k - R\left(x_k + \delta_k\right) \bigg/ \left(1 + \frac{\mu_\delta}{\mu_x}\right)\right)^2. \tag{8}$$

With measurement error in the auxiliary variate, the "mean-of-ratios" estimator in equation (4) becomes

$$\widehat{\tau}_{y2}^* = \overline{r}^*\tau_x^* = \frac{N}{n}\sum_{k\in\mathcal{S}} y_k\left(\frac{\mu_x + \mu_\delta}{x_k + \delta_k}\right),$$

where $\overline{r}^*$ is the average ratio $r_k^* = y_k/x_k^*$ of the n units in the sample. Therefore, $E\left[\widehat{\tau}_{y2}^*\right] = \mu_r^*\tau_x^*$, and

$$B\left[\widehat{\tau}_{y2}^* : \tau_y\right] = \mu_r^*\tau_x^* - \tau_y = \sum_{\mathcal{P}} y_k\left(\frac{\mu_x + \mu_\delta}{x_k + \delta_k} - 1\right), \tag{9}$$

where $\mu_r^* = N^{-1}\sum_{\mathcal{P}} y_k/x_k^*$.

The ratio of the bias of $\widehat{\tau}_{y2}^*$ to $\widehat{\tau}_{y2}$ is

$$\frac{B\left[\widehat{\tau}_{y2}^* : \tau_y\right]}{B\left[\widehat{\tau}_{y2} : \tau_y\right]} = \frac{\sum_{\mathcal{P}} y_k\left(\frac{\mu_x + \mu_\delta}{x_k + \delta_k} - 1\right)}{\sum_{\mathcal{P}} y_k\left(\frac{\mu_x}{x_k} - 1\right)}. \tag{10}$$

The variance of $\widehat{\tau}_{y2}^*$ is

$$V\left[\widehat{\tau}_{y2}^*\right] = \left(\tau_x + N\mu_\delta\right)^2\left(\frac{1}{n} - \frac{1}{N}\right)\sigma_{mr}^{*2}, \tag{11}$$

where $\sigma_{mr}^{*2} = \frac{1}{N-1}\sum_{\mathcal{P}}(\frac{y_k}{x_k^*} - \mu_r^*)^2$.

The error-contaminated version of $\widehat{\tau}_{y3}$ is

$$\begin{aligned}\widehat{\tau}_{y3}^* &= \widehat{\tau}_{y2}^* + \left(\frac{N-1}{N}\right)\left(\frac{n}{n-1}\right)\left(\widehat{\tau}_{y\pi} - \overline{r}^*\,\widehat{\tau}_{x\pi}^*\right)\\ &= \frac{N}{n}\sum_{k\in\mathcal{S}} y_k\left(\frac{\mu_x + \mu_\delta}{x_k + \delta_k}\right)\\ &\quad + \left(\frac{N-1}{N}\right)\left(\frac{n}{n-1}\right)\left(\widehat{\tau}_{y\pi} - \overline{r}^*\,\widehat{\tau}_{x\pi}^*\right),\end{aligned}$$

where $\overline{r}^*$ is the average sample ratio r_k^*, as mentioned earlier. With SRSwoR, $\widehat{\tau}_{x\pi}^* = N(\overline{x} + \overline{\delta})$. Because $E[(\frac{N-1}{N})(\frac{n}{n-1})(\widehat{\tau}_{y\pi} - \overline{r}^*\,\widehat{\tau}_{x\pi}^*)] = \tau_y - \mu_r^*\tau_x^*$, the design unbiasedness of $\widehat{\tau}_{y3}^*$ is preserved despite measurement error in the x_k's.

The variance of $\widehat{\tau}_{y3}^*$ is

$$V\left[\widehat{\tau}_{y3}^*\right] = N^2\left(\frac{1}{n} - \frac{1}{N}\right)\left(\sigma_y^2 + \mu_r^{*2}\sigma_{x^*}^2 - 2\mu_r^* C(x^*, y) + \frac{1}{n-1}\left(\sigma_{r^*}^2\sigma_{x^*}^2 + C(r^*, x^*)^2\right)\right), \tag{12}$$

where $C(x^*, y) = \sum_{\mathcal{P}}\left(x_k^* - \mu_x^*\right)\left(y_k - \mu_y\right)/N$, and $C(r^*, y) = \sum_{\mathcal{P}}\left(r_k^* - \mu_r^*\right)\left(y_k - \mu_y\right)/N$.

3. Empirical Study

The complicated dependence of the bias and variance of $\widehat{\tau}_{y1}^*, \widehat{\tau}_{y2}^*$, and $\widehat{\tau}_{y3}^*$ on the mean and variance of the error-contaminated auxiliary variate prevents a general analytical comparison of the relative performance of these estimators. To circumvent this difficulty we examined the statistical properties of these estimators when sampling from a specific population, which we describe below.

The empirical portion of this study was undertaken to provide an indication of the magnitude of the effects of additive measurement error in x_k on the estimators presented above, and to examine how the magnitude of these effects change with the mean and variance of the measurement error process itself. We imposed various types of measurement error on data collected by Candy (1999). In that study, conducted in Tasmania, Australia, the length, width, and area of *Eucalyptus nitens* leaves were measured. We computed the product of leaf length and width, and used this erstwhile "rectangular" area as the auxiliary variate for the estimation of total leaf area, τ_y, of the population. Descriptive parameters for leaf area and for the corresponding rectangular area are displayed in Table 1. The marginal distribution of leaf area and the relation between leaf area and rectangular area are shown in Figure 1.

4. Measurement Error Processes

We examined the effect of measurement error in the auxiliary variate in the case where the magnitude of the error was constant for all units in the population. In addition, we looked at its effect when the magnitude of measurement error varied among population units in accordance with a

Table 1
Descriptive parameters of the *Eucalyptus nitens* leaves population ($N = 501$)

Descriptive parameter	Variable: Leaf area (cm^2)	Length × width (cm^2)
Minimum	28.1	55.1
Maximum	146.6	222.2
Mean (μ)	71.0	99.8
Standard deviation (σ)	21.1	31.5
Total (τ)	35, 575.2	49, 988.1
Coefficient of variation (γ)	29.7%	31.6%
Coefficient of skewness	0.6	0.9
Kurtosis	0.2	0.7
Correlation coefficient (ρ)	0.96	

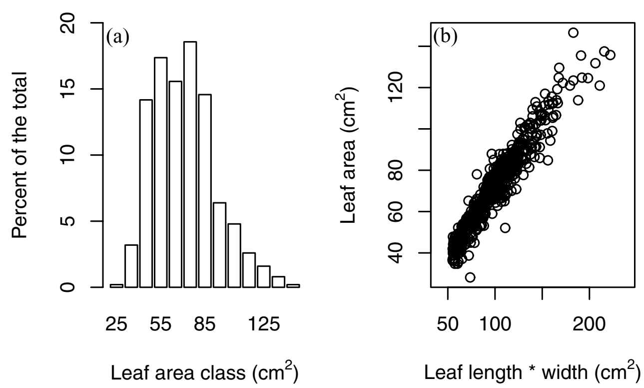

Figure 1. *Eucalyptus nitens* leaves population: (a) histogram of leaf areas and (b) relationship between leaf area and length × width.

uniform distribution, a Gaussian distribution, and a beta distribution.

The case of constant measurement error corresponds to the situation, mentioned above, where instrumentation yields a reading that systematically is larger than it should be, or else is systematically smaller. We examined the effect of such systematic error in the measurement of x_k for the estimation of total *E. nitens* leaf area, τ_y, when its magnitude was $\delta = -25\,cm^2$ for each x_k, i.e., $\delta_k = -25\,cm^2$, $\forall k$. We did likewise when $\delta_k = 25\,cm^2$, $\forall k$, and then at a succession of values on a fine grid within this range.

Uniform measurement error mimics the process of rounding in the recording of x_k. In the empirical study, we generated a uniform random number error, u_k, for each x_k from a $U[-25, 25]$ distribution. We then applied a multiplicative scaling factor, f_U, to each random number, so that the scaled uniformly distributed measurement error was $\delta_k = f_U u_k$. We did this for a minimum value of $f_U = 0$, a maximum value of $f_U = 1$, and at a succession of values on a fine grid within this range. The case where $f_U = 0$ evidently corresponds to the absence of measurement error.

Gaussian measurement error may result from the agglomeration of errors from independent sources. Example sources may include change in ambient temperature, personnel fatigue, altered battery, or signal strength, background noise level, glare, mental distraction, to name a few. We generated δ_k from a $N\ (0,\ \sigma_\delta^2)$ distribution, where $\sigma_\delta = f_N \sigma_x$, and σ_x is the standard deviation of the auxiliary variate in the population of *E. nitens* leaves.

The multiplicative scaling factor, f_N, varied from a minimum of 0.0, a maximum of 0.30, and intermediate values on a fine grid within this range. Here, too, a value of $f_N = 0$ corresponds to the absence of measurement error.

We also wished to examine the effect of skew. For this purpose, we generated δ_k proportional to a beta-distributed random variate, $b_k \sim \beta(a, b)$, with $a = 2$ and $b = 10$. The proportionality factor was set to $f_\beta = 25p/\max\ (b_k)$, where p is a proportion that was varied from a minimum of $p = 0$, a maximum of $p = 1$, and intermediate values on a fine grid within this range. The case where $p = 0$ corresponds to the absence of measurement error. For each set of scaled, beta-distributed measurement errors, δ_k, we deducted the median value from each so that the skewed distribution of measurement errors was centered around zero, as in the uniform and Gaussian cases and the constant measurement error case introduced above.

5. Effect of Constant Measurement Error

In this section, we examine the effect on design-based bias, standard error (SE), and root mean square error (RMSE) caused by the insinuation of a systematic (constant) measurement error in the auxiliary variate. That is, $\delta_k = \mu_\delta$, $\forall k$.

5.1 *Bias Ratio Trend with Change in Average Error of Measurement*

The bias ratio for $\widehat{\tau}_{y1}^*$, equation (6), and for $\widehat{\tau}_{y2}^*$, (10), are displayed in Figure 2a for a range of values of μ_δ arrayed as a proportion of μ_x. Horizontal reference lines have been superimposed at values of -1, 0, and 1 on the vertical axis, as well as a vertical reference line at $\mu_\delta/\mu_x = 0$. As mentioned earlier, these ratios do not depend on sample size, n.

As seen in this graphic, relative to the bias of $\widehat{\tau}_{y1}$ and $\widehat{\tau}_{y2}$, the bias of $\widehat{\tau}_{y1}^*$ and $\widehat{\tau}_{y2}^*$ when $\mu_\delta < 0$ exceeds that of the corresponding estimator in the absence of measurement error. Conversely, positive measurement error results in reduced bias. However, with sufficiently large positive μ_δ, the bias of each estimator becomes negative and larger in absolute magnitude than the bias of the corresponding error-free estimator. Nonetheless, at least for this *E. nitens* leaf population, there is a range, $0 < \mu_\delta \leqslant 0.2\mu_x$, of constant measurement error where the bias is reduced over what it is in the absence of measurement error.

Figure 2b shows the bias of $\widehat{\tau}_{y1}^*$ and $\widehat{\tau}_{y2}^*$, respectively, expressed as a percentage of τ_y. This display serves to emphasize the comparative imperviousness of the bias in $\widehat{\tau}_{y1}$ to this type of measurement error. The results depicted here were computed with equations (5) and (9), presuming a sample of size $n = 7$, which is roughly a 1% sample of the 501 element *E. nitens* leaf population. For larger sample sizes, the bias of $\widehat{\tau}_{y1}^*$ will lay closer to the zero reference line, whereas the percentage bias trend line for $\widehat{\tau}_{y2}^*$ would be unchanged.

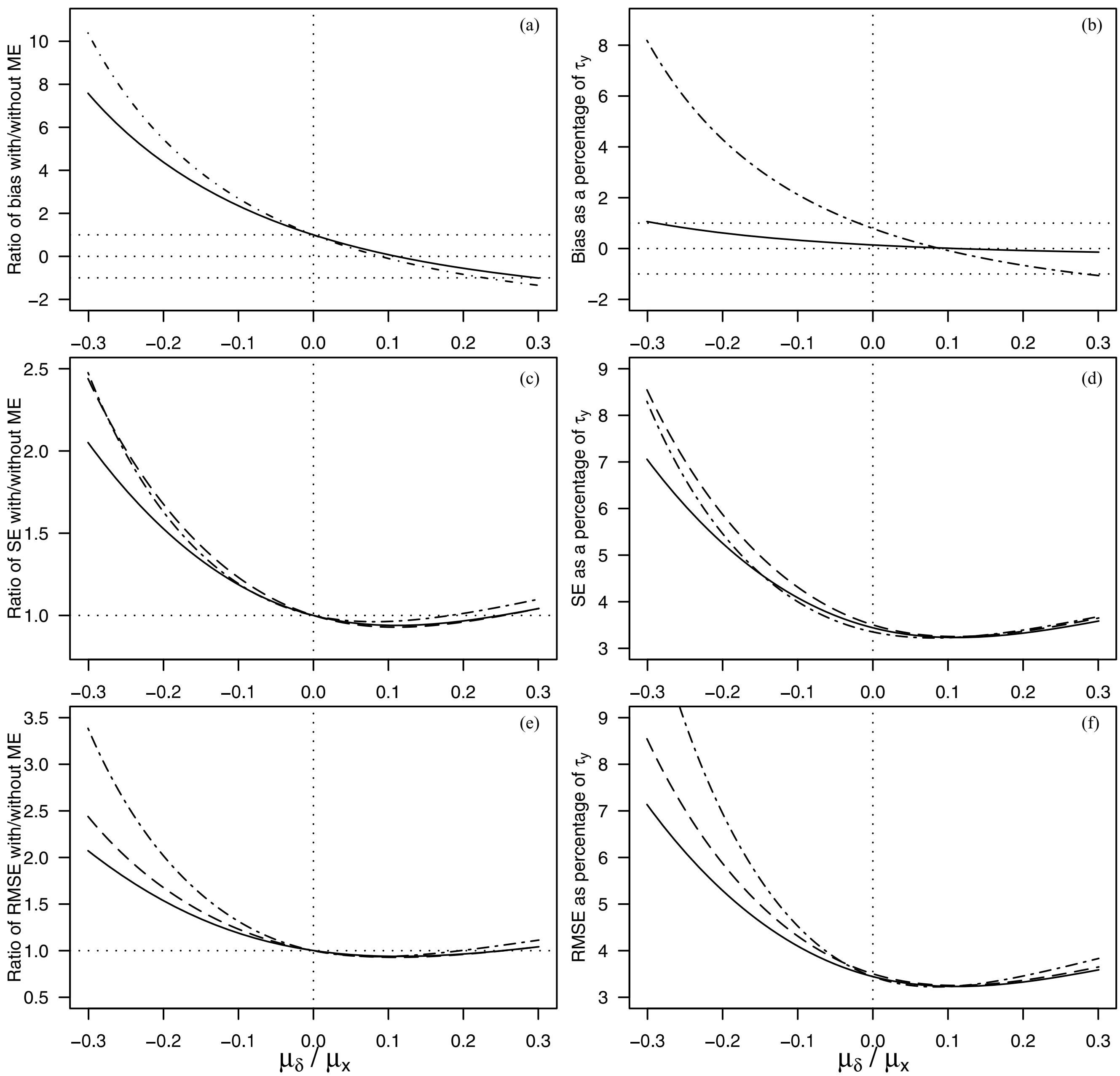

Figure 2. Properties of estimators under systematic measurement error in the x variate. For $\widehat{\tau}^*_{y1}$ (solid line) and $\widehat{\tau}^*_{y2}$ (dot–dash line), the ratio of bias with:without measurement error in (a) and bias as a percentage of τ_y in (b). For $\widehat{\tau}^*_{y1}$, $\widehat{\tau}^*_{y2}$, and $\widehat{\tau}^*_{y3}$ (dashed line), the ratio of standard error with:without measurement error in (c) and standard error as a percentage of τ_y in (d); ratio of RMSE with:without measurement error in (e), and RMSE percentage as a percentage of τ_y in (f). Results for $\widehat{\tau}^*_{y3}$ are based on samples of size $n = 7$.

5.2 *Standard Error Trend with Change in Average Error of Measurement*

The ratio of the standard error of $\widehat{\tau}^*_{y1}$ to $\widehat{\tau}_{y1}$ is displayed in Figure 2c with similar traces for the standard error ratios of $\widehat{\tau}^*_{y2}$ and $\widehat{\tau}^*_{y3}$ superimposed. The standard error ratio of $\widehat{\tau}^*_{y3}$, only, depends on n, and results in this figure are shown for $n = 7$. When $\mu_\delta < 0$, the standard errors of all three estimators are increased, and when $\mu_\delta > 0$, the precision of all three is improved. Although not easily discernible in Figure 2c, when μ_δ is within the region of $\pm 5\%$ of μ_x, the standard error $\widehat{\tau}^*_{y2}$ is less affected than that of $\widehat{\tau}^*_{y1}$, whereas the standard error $\widehat{\tau}^*_{y3}$ is more affected. This is evident, also, when the standard errors of the estimators are expressed as a percentage of τ_y as in Figure 2d. When μ_δ is within the region of $\pm 5\%$ of μ_x, the increase or decrease in standard error is a tiny fraction of a percentage point.

By differentiating equation (8) with respect to μ_δ, we deduce that the standard error of $\widehat{\tau}^*_{y1}$ is minimized at the value of μ_δ satisfying the relation

$$\zeta\mu_\delta = \mu_y - \zeta\mu_x,$$

where $\zeta = C(x, y)/\sigma^2_x$ is the linear regression coefficient of y on x.

5.3 *RMSE Trend with Change in Average Error of Measurement*

A similar set of graphs are shown in Figure 2e, which portrays the RMSE ratios for $\widehat{\tau}^*_{y1}, \widehat{\tau}^*_{y2}$, and $\widehat{\tau}^*_{y3}$, and Figure 2f, which shows the change in RMSE, expressed as a percentage of τ_y, with increasing μ_δ. As before, only the results pertaining to $\widehat{\tau}^*_{y3}$ depend on the size of the sample. In both figures it is apparent that RMSE is affected much more when $\mu_\delta < 0$ than when $\mu_\delta > 0$.

When judged by RMSE, $\widehat{\tau}^*_{y2}$ is most affected when $\mu_\delta < 0$ and $\widehat{\tau}^*_{y1}$ is least affected. When expressed as a percentage of τ_y, as in Figure 2f, $\widehat{\tau}^*_{y1}$ is superior regardless of the sign of μ_δ. All three estimators have smaller RMSE under small positive μ_δ than they do in the absence of measurement error.

6. Effect of Variable Measurement Error

The three distributions—uniform, Gaussian, and beta—of measurement error that we investigated all were centered at zero. By using different scaling factors, we were able to vary the spread of the error distributions in a systematic fashion. In the graphical results discussed in this section, we portray the change in bias, standard error, and RMSE of the error-contaminated estimators of τ_y as a function of the standard deviation, σ_δ, of the error distribution expressed as a proportion of the standard deviation, σ_x, of the auxiliary variate.

The upper panels of Figure 3 display the ratio of the bias of $\widehat{\tau}^*_{y1}$ to $\widehat{\tau}_{y1}$ as σ_δ increases, and the similar ratio of the bias of $\widehat{\tau}^*_{y2}$ to $\widehat{\tau}_{y2}$ is superimposed. From left to right, panels (a), (b), and (c) show the bias ratio, respectively, for uniformly, Gaussian-, and beta-distributed measurement error. In all three cases, the bias of both $\widehat{\tau}^*_{y1}$ and $\widehat{\tau}^*_{y2}$ is increased compared to the bias in the absence of measurement error, becoming greater with increasing dispersion of the error distribution. Under all three error processes $\widehat{\tau}^*_{y2}$ is more sensitive to measurement error than $\widehat{\tau}^*_{y1}$, in the sense that its bias is increased more. For any specified level of σ_δ, the bias ratio was greatest when measurement errors were Gaussian distributed, and least when beta distributed.

The inserts in the upper left corner of these panels show the percentage bias of $\widehat{\tau}^*_{y1}$ and $\widehat{\tau}^*_{y2}$ when $n = 7$. Percentage bias increases in a smooth fashion with increasing σ_δ. Compared to $\widehat{\tau}^*_{y2}$, the bias of $\widehat{\tau}^*_{y1}$ is rather insensitive to increasing measurement error dispersion. At any specified level of σ_δ, bias in $\widehat{\tau}^*_{y1}$ and $\widehat{\tau}^*_{y2}$ was greatest when measurement errors were Gaussian distributed, and least when beta distributed, although the differences between them are slight. Arguably, the salient message carried by these graphs is that variable measurement error increases the bias of $\widehat{\tau}_{y1}$ and $\widehat{\tau}_{y2}$, and that the increase in bias is directly related to the dispersion of the error distribution.

Panels (a), (b), and (c) in the middle row of graphs in Figure 3 show the ratio of standard error with and without measurement error in the auxiliary variate. It is apparent that the standard error of estimation, like bias, increases directly with increasing error dispersion, and that $\widehat{\tau}^*_{y1}$ and $\widehat{\tau}^*_{y3}$ are less sensitive than $\widehat{\tau}^*_{y2}$ in this regard. As with bias, the Gaussian-distributed errors exert more of an effect on the standard error of estimation than uniformly distributed errors, whereas the beta-distributed errors had the smallest effect. This result is evident when examining the standard error of estimation as a percentage of τ_y, shown in the upper left inserts of the middle row of graphs.

When performance is judged by RMSE, $\widehat{\tau}^*_{y1}$ performs best under all three variable measurement error processes, as shown in panels (a), (b), and (c) of the lowest row of graphs of Figure 3.

7. Simulation Results

For the error-contaminated versions of the *E. nitens* leaf population described in Section 3 we checked the results presented in Figures 2 and 3 by means of a simulation study in which we drew 30,000 samples from the population of *E. nitens* leaves contaminated by each of the measurement error processes described in Section 4. In all cases the discrepancy between the results displayed in Figures 2 and 3 and the simulation results agreed to within a fraction of a percentage point. Moreover, when the simulation was repeated with 100,000 samples, the results changed minimally. Sampling was conducted with samples of size 7, 15, and 37. Because of the close similarity of results, we report results for samples of size 7 only.

Aside from serving this confirmatory purpose, the simulation enabled us to evaluate how well the approximations put forth in equations (7), (11), and (12) portray the variance of $\widehat{\tau}^*_{y1}, \widehat{\tau}^*_{y2}$, and $\widehat{\tau}^*_{y3}$, respectively, under the different types of measurement error processes we examined. Cochran (1977, p. 162–163) considered their adequacy in the absence of measurement error.

Results for the case of constant measurement error in the auxiliary variate are displayed in Table 2a. In the absence of measurement error, i.e., when $\mu_\delta = 0$, the standard errors computed from these variance approximations all are within 1% of the Monte Carlo standard errors for all three estimators of τ_y and for the three sample sizes that we examined. When $\mu_\delta > 0$, the computed standard error of $\widehat{\tau}^*_{y1}$ appears to track the Monte Carlo standard error as well as it does when $\mu_\delta = 0$: there is no apparent pattern of increasing or decreasing deviation from the Monte Carlo error. The same can be said for the deviation of the standard error of $\widehat{\tau}^*_{y2}$ and $\widehat{\tau}^*_{y3}$ from the empirical error observed in the simulation study.

When $\mu_\delta < 0$, these approximations to the standard error of estimation perform less well, especially for the largest n. However, only in one instance did the deviation from the empirical standard error exceed 2% in absolute magnitude, and that case occurred only when μ_δ was -25% of μ_x. Overall, the variance approximations given for all three estimators lead to trustworthy standard errors of estimation under constant measurement error in the auxiliary variate.

Results for variable measurement error are tabulated in Table 2b. Recall that we examined the effects of uniform,

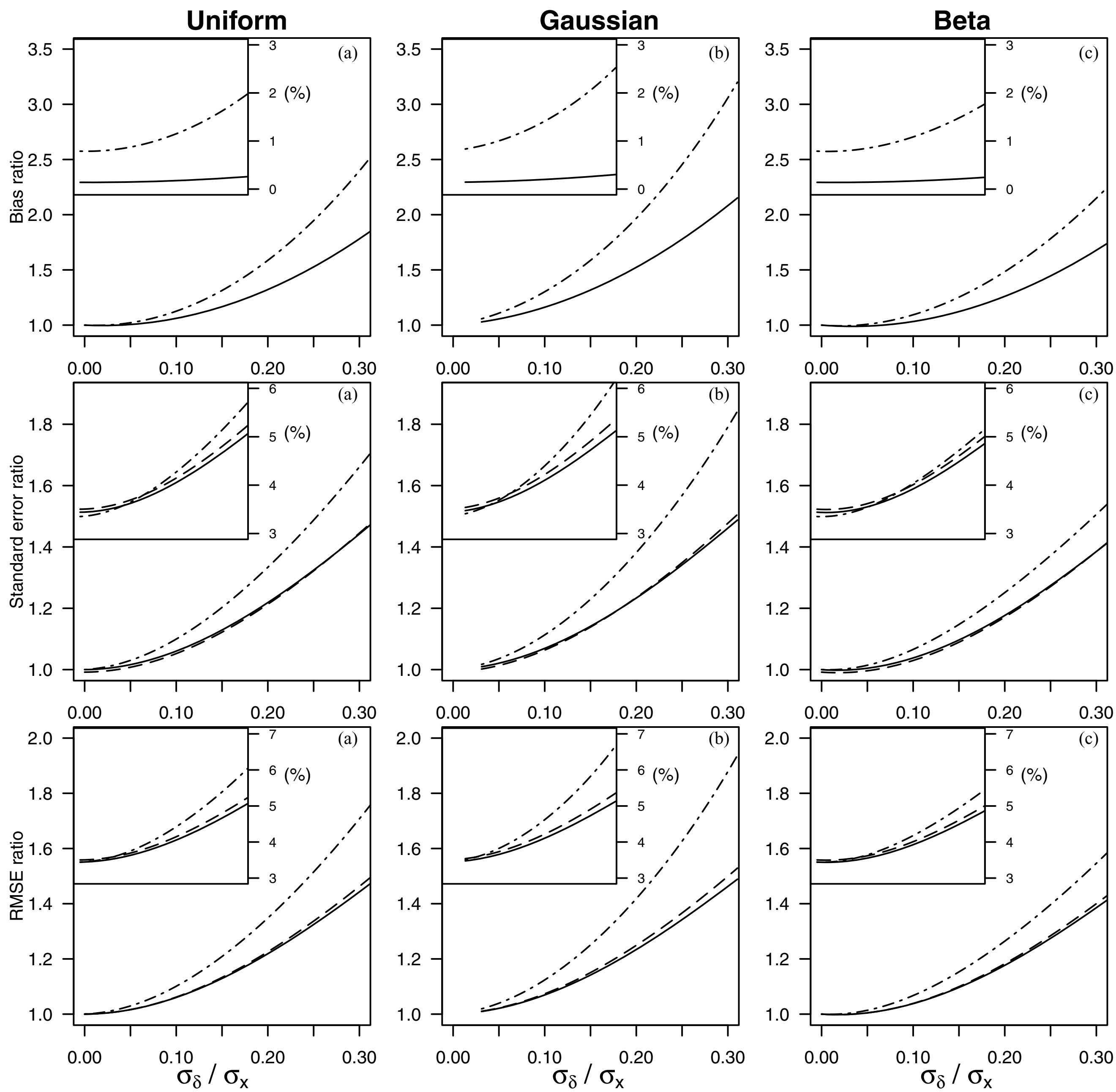

Figure 3. Bias (first panel row), standard error (second panel row), and RMSE (third panel row) of $\widehat{\tau}^*_{y1}$ (solid line), $\widehat{\tau}^*_{y2}$ (dot–dash line), and $\widehat{\tau}^*_{y3}$ (dashed line) relative to that of $\widehat{\tau}_{y1}, \widehat{\tau}_{y2}$, and $\widehat{\tau}_{y3}$, respectively, with uniform (a), Gaussian (b), and beta (c) distributed measurement error in the auxiliary variate. The inner plots represent bias (first panel row), standard error (second panel row), and RMSE (third panel row) expressed as a percentage of τ_y. The horizontal axis of the inserts span the range $0 \leqslant \sigma_\delta/\sigma_x \leqslant 0.30$. Results for $\widehat{\tau}^*_{y3}$ are based on samples of size $n = 7$.

Gaussian, and beta-distributed measurement error, where, in each case, the distribution was centered at zero, i.e., $\mu_\delta = 0$.

Looking first at the results that pertain to uniformly distributed measurement error, there is an apparent pattern of understatement of actual standard error, which increases in magnitude with increasing σ_δ. When $\sigma_\delta = 0.3\sigma_x$, the computed standard errors of $\widehat{\tau}^*_{y1}$, $\widehat{\tau}^*_{y2}$, and $\widehat{\tau}^*_{y3}$ are about 2–4% smaller than what was observed among the 30,000 estimates observed in the simulation study.

In contrast, when $\delta \sim N[0, \sigma^2_\delta]$, the computed approximations to the standard errors of $\widehat{\tau}^*_{y2}$ and $\widehat{\tau}^*_{y3}$ tend to overstate the empirically observed standard error, with the overstatement increasing moderately with increasing σ_δ. This trend was apparent for $\widehat{\tau}^*_{y1}$ at the large sample sizes ($n = 15$ & 37) that we examined.

Results when δ is distributed as a beta random variable mimic the Gaussian results, but the overstatement is greater for a similar level of σ_δ. Whereas overstatement ranged

Table 2

Percentage deviation of the standard error approximation of $\widehat{\tau}^*_{y1}$, $\widehat{\tau}^*_{y2}$, and $\widehat{\tau}^*_{y3}$ from the Monte Carlo standard error. In part (a), results are shown when the measurement error in the auxiliary variate is a constant magnitude given by μ_δ; in part (b), results are shown when the measurement error in the auxiliary variate varies according to uniform, Gaussian, and β distributions with dispersion indicated by σ_δ. Simulation results are based on 30,000 samples of size $n = 7$.

(a)		μ_δ/μ_x										
Distribution	Estimator	−0.25	−0.20	−0.15	−0.10	−0.05	0	0.05	0.10	0.15	0.20	0.25
Constant	$\widehat{\tau}^*_{y1}$	−3.05	−1.90	−0.94	−0.19	0.34	0.62	0.67	0.55	0.36	0.15	−0.03
	$\widehat{\tau}^*_{y2}$	−0.74	−0.61	−0.43	−0.20	0.05	0.28	0.45	0.54	0.56	0.53	0.48
	$\widehat{\tau}^*_{y3}$	−0.85	−0.76	−0.63	−0.46	−0.24	−0.02	0.19	0.34	0.42	0.43	0.41
(b)		σ_δ/σ_x										
		0	0.03	0.07	0.10	0.13	0.17	0.20	0.23	0.27	0.30	
Uniform												
	$\widehat{\tau}^*_{y1}$	0.92	0.83	0.49	−0.03	−0.64	−1.89	−2.47	−3.02	−3.54	−4.04	
	$\widehat{\tau}^*_{y2}$	0.44	1.01	1.19	1.03	0.64	−0.43	−0.99	−1.55	−2.10	−2.65	
	$\widehat{\tau}^*_{y3}$	0.29	0.26	0.13	−0.08	−0.34	−0.92	−1.21	−1.48	−1.75	−2.02	
Gaussian												
	$\widehat{\tau}^*_{y1}$	0.49	0.47	0.47	0.46	0.46	0.45	0.43	0.40	0.35	0.29	
	$\widehat{\tau}^*_{y2}$	−0.31	0.03	0.36	0.67	0.96	1.23	1.50	1.77	2.09	2.47	
	$\widehat{\tau}^*_{y3}$	−0.05	0.02	0.17	0.39	0.66	0.95	1.25	1.55	1.83	2.09	
Beta												
	$\widehat{\tau}^*_{y1}$	0.76	1.22	1.60	1.88	2.07	2.21	2.20	2.16	2.09	2.00	
	$\widehat{\tau}^*_{y2}$	0.26	0.83	1.47	2.07	2.57	3.21	3.35	3.41	3.40	3.32	
	$\widehat{\tau}^*_{y3}$	0.21	0.75	1.28	1.76	2.18	2.77	2.97	3.10	3.19	3.25	

generally from 0–2% when $\sigma_\delta = 0.3\sigma_x$ under Gaussian measurement error, it ranged from 2–3% under beta measurement error.

8. Discussion

This has been a multifaceted study, the major conclusions of which are enumerated below. Although it is impossible to generalize the results of a single empirical study, our results at the very least provide insight into how the effects of measurement error change with increasing μ_δ in the case of constant measurement error and on increasing measurement error dispersion in the case of variable measurement error. The analytical expressions for the bias and variance of $\widehat{\tau}^*_{y1}, \widehat{\tau}^*_{y2}$, and $\widehat{\tau}^*_{y3}$ have been presented in Section 2.3 and confirmed by the simulation study.

(i) The unbiasedness of $\widehat{\tau}_{y3}$ is preserved even when measurement error contaminates the auxiliary variate. As a referee pointed out, this estimator is unbiased without regard to the auxiliary variate and how or whether it may be contaminated.

(ii) Constant measurement error in the auxiliary variate can occur when the measuring instrument is faulty in a consistent manner. The effects of such errors are not symmetric: when $\mu_\delta < 0$, its effect on the bias, standard errors, and mean square errors of $\widehat{\tau}^*_{y1}$ and $\widehat{\tau}^*_{y2}$ differ from its effect when $\mu_\delta > 0$. The same can be said regarding its effect on the standard error of $\widehat{\tau}^*_{y3}$. When $\mu_\delta > 0$, the bias and standard error are both reduced from their values in the absence of measurement error. It will be useful to determine whether this result holds for other study populations. Over a broad range of $\mu_\delta \neq 0, \widehat{\tau}^*_{y1}$ has smallest RMSE.

(iii) Variable measurement error in the auxiliary variate can occur for any of a number of reasons, e.g., rounding error in the measurement process. In the face of variable measurement error, the bias of $\widehat{\tau}^*_{y1}$ increases less than that of $\widehat{\tau}^*_{y2}$. The RMSEs of all three estimators we examined increases directly with increasing σ_δ, a result which is intuitively clear. The RMSEs of $\widehat{\tau}^*_{y1}$ and $\widehat{\tau}^*_{y3}$ are nearly identical, both being less than that of $\widehat{\tau}^*_{y2}$.

(iv) The approximations we presented to the variance, and hence standard error, of the three estimators work well under constant measurement error when $\mu_\delta > 0$, but deteriorate slightly when $\mu_\delta < 0$. Under a variable measurement error process, the magnitude and direction of its effect on the standard error approximation differ from one error distribution to another. Overall, the absolute size of the effect increases directly with increasing σ_δ.

There are a number of ways in which the results presented here can be extended. It would be of interest to determine how estimators of variance are affected by measurement error in the auxiliary variate, as well as the coverage of nominal $(1 - \alpha)100\%$ confidence intervals are affected by measurement error in the auxiliary variate. When the relationship between y and x does not pass near the origin, the linear regression estimator is more apt, in the model-assisted sense of Särndal et al. (1992). It is not clear whether the results of measurement error in x observed for the ratio estimators of this article would be magnified or diminished with the regression estimator of τ_y. Although we have taken a design-based approach to infer the effect of fixed measurement error, there may be utility in a model-based approach that postulates a probability distribution for the error of repeated measurement of each unit of the population. Surely for some practitioners who are more accustomed to relying on a presumed model as the basis for statistical inference, this would be a more satisfying approach. Accordingly, we shall report on our study of this approach later. Lastly, the combination of measurement error in both y and x, possibly stemming from different error processes, needs to be examined.

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